ON THE EMISSION (RADIATION) OF AN ELASTIC WAVE FROM A SPHERICAL EXPLOSION IN THE GROUND

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PMM Vol.24, No.1, 1960, pp. 126-133

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(Received 12 October 1959)

A further development of the work [1] is presented here. In comparison with the paper mentioned, this article contains a study of an elastic-plastic medium (depending on the effect of compaction) rather than a rigid-plastic medium, while the flow of the material in the incompressible state achieves a property similar to internal friction. The presence of an initial elastic component in the (σ, θ) diagram allows us to study the entire process of the propagation of the shock wave including the emission (radiation) of an elastic wave. This problem is investigated below as having spherical symmetry.

1. The properties of the medium should be given by means of a law of volume change and a law of deformation due to change of shape. The law of volume change (σ – θ diagram) is given in the form of a broken line (Fig. 1). It is assumed that the medium can exist only in two states: the elastic initial (segment OA) and the compacted incompressible (B_1BC) state; the transition to the second state takes place instantaneously at $\sigma = \sigma_s$ (this assumption requires, in general, a proof). σ denotes the mean stress and θ denotes the dilatation. Line $OABC$ corresponds to the active state (loading state). The unloading in the elastic state represents a reversible process, while in the inelastic state the unloading takes place with a change of volume (straight line CBB_1). In the elastic state (line OA) the change of shape is subject to Hooke's law. During flow in the incompressible state plastic work is expended. It is necessary to make some assumptions about what kind of work this is. In the plastic flow of metals the influence of the mean stress upon the plastic work is insignificant. Such an assumption appears not to be applicable

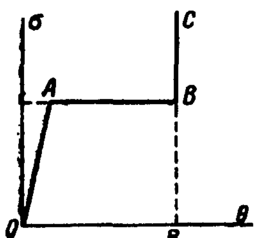

FIG. 1.

for soft soil. In this latter case it is more plausible to assume that the plastic work increases with an increase of the mean stress. So far, there have been few experiments along these lines. I propose to assume as a hypothesis (which should be verified by means of an experiment) that for spherical symmetry the change of the elemental plastic work in the transition from one state to a neighboring one be proportional to the change of the largest shear. The proportionality coefficient is assumed to be a function of the mean stress. This function should be determined by experiment. The mathematical formulation of this hypothesis is

$$\delta A = \iiint m(\sigma)\,|\delta\gamma|\,dV \tag{1.1}$$

Here A is the plastic work, δA is its change during the transition to the neighboring state, $\delta\gamma$ is the greatest shear during this transition. As long as there is no experimental information on the function $m(\sigma)$, one has to make simple expedient assumptions. In the present article I shall assume that $m(\sigma)$ is a linear function.*

This assumption leads to a plastic medium proposed at first by Ishlinskii [2]. A similar assumption on the soil property was made by Kompaneets in [3]. The following is a study of the propagation of a spherical wave in the described continuous medium caused by the detonation of a charge which filled a spherical cavity (cavern). It is assumed that at $t = 0$ the explosives occupy the initial volume of the cavity. The expansion is accomplished adiabatically. We do not analyze the wave process inside the cavity.

2. We relate the motion to a spherical system of coordinates whose center is located at the center of the spherical cavity. We denote the running radius by r, let $u(r, t)$ be the radial displacement, $v(r, t)$ the radial velocity, σ_r σ_α σ_β the stress components. Under the conditions of spherical symmetry $\sigma_\beta = \sigma_\alpha$ and we obtain for the mean stress

$$\sigma = \frac{1}{3}(\sigma_r + 2\sigma_\alpha)$$

In the elastic as well as in the plastic state the equation of motion has the form

$$\frac{\partial\sigma_r}{\partial r} + \frac{2(\sigma_r - \sigma_\alpha)}{r} = \rho_{1,2}\left(\frac{\partial v}{\partial t} + v\frac{dv}{\partial r}\right) \tag{2.1}$$

For an elastic medium $\rho_{1,2} = \rho_1$. It is necessary to supplement Equation (2.1) by Hooke's law

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* A soil model of very general form was proposed by S.S. Grigorian [4]. The scheme studied in this paper is a special case of it.

$$\sigma_r = \lambda\theta + 2\mu\frac{\partial u}{\partial r}, \qquad \sigma_\alpha = \lambda\theta + 2\mu\frac{u}{\partial r} \qquad \left(\theta = \frac{\partial u}{\partial r} + \frac{2u}{r}\right)$$

In the elastic state, as usual, we do not differentiate between the Eulerian and the Lagrangean description of the process, i.e. $\partial u/\partial t = dv/dt$.

In the plastic state, because of incompressibility, we have

$$v = \frac{C(t)}{r^2}$$

where $C(t)$ is an arbitrary function. Assuming that then

$$\frac{\partial\gamma}{\partial t} = \frac{\partial v}{\partial r} - \frac{v}{r} = -\frac{3C(t)}{r^3}$$

and analyzing the plastic flow of an elemental spherical layer, it follows on the basis of the energy balance that

$$\frac{\partial\sigma_r}{\partial r} + \frac{3m(\sigma)}{r} = \rho_2\frac{dv}{dt} \tag{2.2}$$

Substituting from (2.1) we find

$$\sigma_r - \sigma_\alpha = \frac{3}{2}m(\sigma) \tag{2.3}$$

which represents some "plasticity condition". As long as the necessary experimental results are not available we shall propose that

$$\tfrac{3}{2}m(\sigma) = m_0\sigma + m_1$$

From this follows

$$\sigma_\alpha = \frac{(1 - m_0')\sigma_r - m_1}{1 + 2m_0'}, \qquad m_0' = \frac{m_0}{3}$$

In this paper we choose for m_0' the value 1/4. This leads to the plausible result

$$\sigma_\alpha = \tfrac{1}{2}\sigma_r - \tfrac{2}{3}m_1 \tag{2.4}$$

The constant m_1 has the dimensions of stress; it is expedient to assume that $m_1 > 0$.

Let us assume an adiabatic expansion of the explosion products. Thus we find the stress σ_0 at the walls of the cavity to be

$$\sigma_0(r_0) = \sigma_0(a)\left(\frac{a}{r_0}\right)^{3\gamma} \tag{2.5}$$

where a is the initial radius of the cavity and γ the adiabatic exponent.

It is assumed that the initial stress $\sigma_* = \sigma_0(a)$ is sufficiently large and capable of causing in the vicinity of the cavity a compaction of the soil which then continues to propagate. With the hypotheses stated above the propagation process of the shock wave can be described in terms of the following consecutive stages:

1) the compaction shock wave propagates through the undisturbed medium;

2) an elastic wave propagates through the undisturbed medium; behind it there follows a compaction zone whose boundary is the shock front, and the compaction of the soil continues;

3) an elastic wave propagates through the undisturbed medium; behind it there follows a compacted zone whose boundary is a contact discontinuity. The plastic flow continues, but there is no new compaction of the medium;

4) a compacted zone is established; the back front of the elastic wave has broken away from it and goes to infinity.

3. Let us set up the equations and boundary conditions for the description of each of the enumerated stages (the question of uniqueness of the solution of the problem has not yet been studied).

Let us analyze the first stage (Fig. 2). The radius of the shock wave is denoted by r_s. The mechanical conditions at the shock wave reduce to the following two:

$$v_2(r_s, t) = \alpha r_s', \qquad \sigma_2(r_s, t) = -\alpha\rho_1 r_s'^2, \qquad \alpha = 1 - \frac{\rho_1}{\rho_2} \tag{3.1}$$

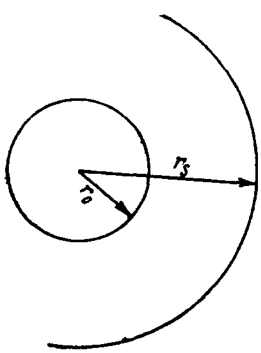

FIG. 2.

The index 2 refers to the compacted state of the soil, the index 1 to the elastic state. The prime indicates differentiation with respect to time.

In the description we have adopted the soil represents a medium with a separating energy. Thus the mechanical conditions at the shock wave are not related to the energy conditions, which should serve as a check for the energy balance at the shock wave.

The equation of motion (2.1) together with the incompressibility condition and the plasticity condition (2.3) yields the following expression for the radial stress component:

$$\sigma_r = \rho_2 C'(t)\frac{\ln r}{r} + \frac{2\rho_2}{3}\frac{C^2(t)}{r^4} + \frac{C_1(t)}{r} - m' \qquad \left(m' = \frac{4}{3}m_1\right) \tag{3.2}$$

Here $C(t)$ and $C_1(t)$ are arbitrary functions of time. Functions $C(t)$ and $C_1(t)$, just as two other functions $r_0(t)$ and $r_s(t)$, should be found from the boundary conditions. Two conditions (3.1) at the shock wave are already written out; the two other conditions are stated at the moving boundary of the cavity

$$v(r_0, t) = r_0'(t), \qquad \sigma(r_0, t) = \sigma_0(r_0) \tag{3.3}$$

One of the four unknown functions can be taken as the basic unknown for which, by eliminating the remaining ones, we obtain one equation. We choose r_s^3 as the unknown and denote this quantity by the letter z. We obtain an ordinary second order differential equation for z, which is nonlinear and which does not contain time explicitly. If one interchanges the roles of the unknowns by taking z as the independent variable at $z' = dz/dt$ at the unknown function then we obtain for z'^2 a first order linear equation

$$\frac{dz'^2}{dz} + P(z)z'^2 = Q(z) \tag{3.4}$$

Here

$$P(z) = \frac{2\rho_1}{3\rho_2}\frac{1 + \dfrac{2a^3}{\alpha z + \beta}}{z\ln\dfrac{z}{\alpha z + \beta}}, \qquad Q(z) = \frac{18}{\alpha\rho_2}\frac{m' z^{1/3} - (\alpha z + \beta)^{1/3}(\sigma_0 + m')}{\ln\dfrac{z}{\alpha z + \beta}}$$

Equation (3.4) is to be integrated over the semi-infinite segment $(a^3, +\infty)$. The initial condition (at $z = a^3$) is obtained if in the other equations (3.1) and (3.3) one goes over to the limit, letting $r_0 \to a$ and $r_s \to a$.

Thus, we obtain

$$z'^2(a^3) = -\frac{9a^4}{\alpha\rho_1}\sigma_m \tag{3.5}$$

The function σ_0, which enters $Q(z)$, depends on z and, according to (2.5), is expressed by the formula

$$\sigma_0(r_0) = \sigma_m\left(\frac{a^3}{\alpha z + \beta}\right)^{\gamma} \tag{3.6}$$

Equation (3.4) is integrated by quadratures; an evaluation of the indefinite integrals in terms of elementary functions is not performed. The integrals can be evaluated approximately by different methods. However, before one goes into approximate numerical calculations, it is necessary to determine the qualitative properties of Equation (3.4).

When studying the field of directions determined by this equation, one can show that the solution of Equation (3.4), which corresponds to the initial condition (3.5), decreases monitonically with increasing z (at least, starting from some value of z). Furthermore, evaluating $|Q(z)|$ one can show that $\lim z'^2 = 0$ as $z \to \infty$. This means that the velocity of motion of the shock wave front decreases and approaches zero. Such a derivation leads to the conclusion that the first phase of motion cannot continue for all time. After the velocity of the shock wave becomes equal, and later also less than the speed of sound in the undisturbed medium, there should appear an elastic (sound) wave in front of the shock wave, i.e. there appears a second phase of motion.

4. The new phase of motion will differ by the conditions at the shock wave, which now propagates along the zone of elastic disturbance.

Instead of conditions (3.1) we obtain

$$\sigma_2(r_s, t) = \sigma_1(r_s, t) - \frac{\alpha\rho_2^2}{\rho_1}[r_s' - v_2(r_s, t)]^2 \tag{4.1}$$

$$v_2(r_s, t) = v_1(r_s, t)\frac{\rho_1}{\rho_2} + \alpha r_s' \tag{4.2}$$

In the plastic zone the stress σ_r is expressed as before by formula (3.2).

The conditions (3.3) at the boundary of the cavity are also preserved.

In the elastic zone the radial stresses and the velocity are expressed, respectively, by the formulas

$$\sigma_r = \frac{\lambda + 2\mu}{a_0^2 r} F''\left(t - \frac{r}{a_0}\right) + \frac{4\mu}{a_0 r^2} F'\left(t - \frac{r}{a_0}\right) + \frac{4\mu}{r^3} F\left(t - \frac{r}{a_0}\right) \tag{4.3}$$

$$v = -\frac{1}{a_0 r} F''\left(t - \frac{r}{a_0}\right) - \frac{1}{r^2} F'\left(t - \frac{r}{a_0}\right) \tag{4.4}$$

where a_0 is the speed of sound in the elastic medium.

The forward front of the elastic zone does not generate any additional conditions. The only requirement is its propagation "along the characteristic," i.e. with sound velocity.

Summing up what we have said so far about the problem of the second phase of motion, we come to the conclusion that it is necessary here to determine five unknown functions $C(t)$, $C_1(t)$, $r_0(t)$, $r_s(t)$ and $F(t)$, while there are only four equations (4.2), (4.1) and (3.3) available. The additional condition can be obtained from considerations of the stability of the shock wave. It is known that independently from the thermodynamic properties of the medium [5] the shock wave propagates with supersonic velocity along the region which lies ahead of its front and with subsonic velocity along the region extending behind the front. Behind the front the medium is incompressible, the speed of the propagation of sound is infinite and therefore the corresponding requirement is always fulfilled. From the point of view of the second phase of motion, in this phase the velocity of the shock wave with respect to the particles lying ahead of the front is always smaller than the velocity of sound in the region ahead of the front. This means that for $0 < \sigma_1 < \sigma_s$ the motion will be definitely unstable. If a motion nevertheless takes place in the second phase then this will be possible only on one assumption, namely, that

$$\sigma_1 = \sigma_s \tag{4.5}$$

This assumption we shall treat, in fact, as the missing condition (a direct proof of the stability of motion in this case should also be performed)*.

Let us also make the following simplification: the second phase occurs when the radius of the front of the shock wave already considerably exceeds (several times) the initial radius of the cavity. Then it is natural to replace formulas (4.3) and (4.4) by the following approximation:**

$$\sigma_r \approx \frac{\lambda + 2\mu}{a_0^2 r} F''\left(t - \frac{r}{a_0}\right), \qquad v \approx -\frac{1}{a_0 r} F''\left(t - \frac{r}{a_0}\right) \tag{4.6}$$

From the assumption (4.5) considering

$$\frac{\sigma_r}{\sigma} = \frac{\lambda + 2\mu}{\lambda + {}^2/_3\mu}$$

it follows that the boundary values σ_r and v at the front of the shock wave at the side of the elastic region will be

$$\sigma_1(r_s, t) = \frac{\lambda + 2\mu}{\lambda + {}^2/_3\mu}\sigma_s, \qquad v_1(r_s, t) = \frac{a_0 \sigma_s}{\lambda + 2\mu} \tag{4.7}$$

Since now the boundary conditions ahead of the shock wave front are known, the four equations (4.2), (4.1) and (3.3) are sufficient to determine the four functions $C(t)$, $C_1(t)$, $r_0(t)$ and $r_s(t)$. The unknown function $F''(t)$ is then found from the functional equation

$$F''\left(t - \frac{r_s(t)}{a_0}\right) = \frac{a_0^2 \sigma_s}{\lambda + {}^2/_3\mu} r_s(t) \tag{4.8}$$

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* See also [6].

** This simplification is the more correct the smaller the extent of the second phase of motion. It would be desirable to replace this assumption by more accurate ones in future.

Conditions (3.3) and (4.2) are explicitly

$$C(t) = r_0^2 r_0', \qquad C(t) = \alpha r_s^2 r_s' + \frac{\rho_1}{\rho_2} v_1$$

From that

$$d(r_0^3) = \left(\alpha + \frac{\rho_1}{\rho_2} \frac{v_1}{r_s'}\right) d(r_s^3) \tag{4.9}$$

At the beginning of the second phase v_1/r_s' is a very small quantity, and at the end it approaches unity. As a first approximation we take

$$d(r_0^3) = \alpha d\,(r_s^3) \tag{4.10}$$

This is justified by the consideration that in the expansion of the cavity the main part should be played by the compaction of the soil, while the contribution to this expansion of the elastic yielding of the external region cannot be considerable. (This assumption refers to a disguised explosion. With the participation of a free surface the mechanism of the expansion of the cavity may be some other one).

The integration of Equation (4.10), taking into account the initial condition, which requires continuity of r_0 as a function of r_s, leads to the result

$$r_0^3 = \alpha r_s^3 + \beta$$

Now one can introduce, as in the previous section, $z = r_s^3$ as the independent variable and z'^2 as the unknown function. We obtain the following equation for this function

$$\frac{dz'^2}{dz} \ln \frac{z}{z_0} + \frac{2\rho_1}{3\rho_2} \frac{z'^2}{z} \left(1 + \frac{2a^3}{z_0}\right) = \frac{18}{\alpha\rho_2} \left(\frac{\lambda + 2\mu}{\lambda + {}^2/_3\mu} \sigma_s + m'\right) z^{1/3} - \frac{18}{\alpha\rho_2} [\sigma_0 (r_0) + m'] z_0^{1/3} -$$
$$- \frac{4\rho_1}{\alpha\rho_2} v_1 \frac{z'}{z^{1/3}} \left[\ln \frac{z}{z_0} - \alpha - 2\alpha \left(\frac{z}{z_0}\right)\right] - \frac{18\rho_1}{\alpha\rho_2} v_1^2 z^{1/3} \left[\alpha + \frac{2\rho_1}{3\rho_2}\left(1 - \frac{z}{z_0}\right)\right] \tag{4.11}$$

Here $z_0 = \alpha z + \beta$ and function σ_0 is expressed by means of formula (3.6). This first order equation is nonlinear in z'^2. The initial condition is the continuity condition on z' at the transition from the first phase of motion to the second one. The continuity of z' follows from the fact that although the right-hand sides of the differential equations (3.4) and (4.11) are different, the changes, however, do not have an "impulsive" character during the change of the phases of motion. One can construct isoclinics for Equation (4.11) such that one can clarify the qualitative character of the motion in the second phase. The study shows that in this phase z' (and consequently also the velocity of the front r_s') will decrease monotonically and approach zero.

5. It is easily shown that the described phase of motion cannot extend to the state of test. Actually, before r_s goes to zero, the equality $r_s = v_1$ will occur, and because of the law of conservation of mass

$$(r_s' - v_1)\rho_1 = (r_s' - v_2)\rho_2$$

where $r_s' = v_2$ should hold. Thus at the given moment the shock wave, having completely exhausted itself, ceases to exist. However, the motion still persists. It is natural to describe the following, third, stage of motion as a motion with a contact discontinuity, where at the boundary, which separates the compacted region from the elastic one, the velocities as well as the stresses are equal.

The boundary conditions at the contact discontinuity will then be expressed as follows:

$$\sigma_2(r_s, t) = \sigma_1(r_s, t), \qquad v_2(r_s, t) = v_1(r_s, t) \tag{5.1}$$

One should supplement them by the condition due to which the surface of the contact discontinuity would consist of identically the same particles

$$r_s' = v_1(r_s, t) \tag{5.2}$$

The conditions (3.3) at the boundary of the cavity remain the same as before. Thus we have here five relations for determining the values

$$C(t), \quad C_1(t), \quad r_0(t), \quad r_s(t), \quad v_1(r_s, t), \quad \sigma_1(r_s, t)$$

The last two functions v_1 and σ_1 are independent. Actually, within the accepted accuracy

$$\sigma_1(r_s, t) = -(\lambda + 2\mu)\frac{v_1(r_s, t)}{a_0}$$

The number of equations corresponds to the number of unknowns which can thus be found from here. If we introduce as before $z = r_s^3$, we obtain a differential equation for this unknown function

$$\frac{dz'^2}{dz}\ln\frac{z}{z-\beta_0} - \frac{4\beta_0 z'^2}{3z(z-\beta_0)} = -\frac{18}{\rho_2}\sigma_m\frac{a^{3\gamma}}{(z-\beta_0)^{\gamma-1/3}} -$$

$$\frac{6(\lambda+2\mu)}{a_0\rho_2}\frac{z'}{z^{1/3}} - \frac{18m'}{\rho_2}[z^{1/3} - (z-\beta_0)^{1/3}] \tag{5.3}$$

Here β_0 is the volume of the medium compacted during the whole duration of the process multiplied by $3/4\,\pi$. The initial condition on z'^2 comes from the continuity of this function at the transition from the second phase of motion to the third one.

A study of Equation (5.3) shows that z'^2 goes to zero at some finite value z. This corresponds to the stopping of the motion of the plastic layer. There the third stage of motion ends.

After z'^2 has been found as a function of z one can determine the law of motion of the surface of discontinuity (impulsive or contact). In fact, if

$$z'^2 = f(z), \quad \text{then} \quad \int_{a'}^{z} \frac{d\zeta}{\sqrt{f(\zeta)}} = t$$

under the condition that the time is counted from the instant of the explosion. It should be noted that, although in principle this third phase of motion is unavoidable, its duration and the practical meaning are obviously negligible.

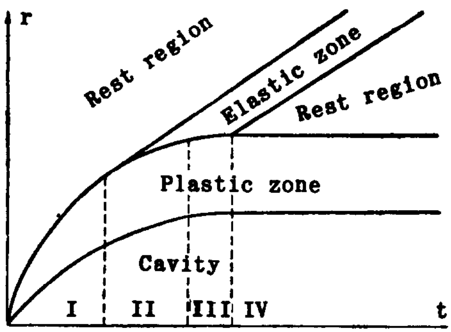

FIG. 3.

6. The last stage of motion, which arises after the stopping of the plastic zone, remains to be studied. In this stage the motion takes place only along the outside region where the elastic wave propagates. Its propagation is determined by that "initial" velocity distribution which occurs at the instant of the stopping of the plastic zone. There is then a rear front being formed next to it, which separates it from the boundary of the compressed zone, and the elastic wave goes to infinity. This part of the problem is solved by well-known means.

From our analysis in relation to the accepted values of the parameters, we can determine the instances of transition of one phase of motion into the other, the radius and volume of the compacted zone, the radius of the cavity, the energy of the radiated elastic wave and the energy irreversibly lost in plastic deformation. However, the solution of all these problems requires the completion of numerical calculations. The kinematical picture of the propagation of the boundary of the cavity and the boundary of the compressed medium can be portrayed qualitatively in Fig. 3, where time is plotted along the abscissa, and the boundaries of the respective regions during all studied phases of motion are plotted along the ordinate.

I should like to express my gratitude to A.A. Grib and S.S. Grigorian for discussing this paper with me.

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Translated by M.I.Y.